How to Write Maxwell's Equations on a T-Shirt

At MIT, Caltech, and other trade schools, one often sees engineers wearing t-shirts that read:

And God said:

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \qquad \vec{\nabla} \times \vec{H} = \frac{\partial}{\partial t}\vec{D} + \vec{j} \tag{1}$$

and there was light!

I do not claim any special revelation or theological knowledge, but it seems to me unlikely that God really said this. The fields E and B can make some claim to fundamental importance, since their sources are elementary charges and currents, and since they appear in Lorentz-covariant wave equations. However, D and H are on a different level; they are combinations of E and B together with the grubby details of a particular medium, expressed though the local electric and magnetic dipole densities. I doubt that God would have included these effects explicitly in His equations, when they appear implicitly in the more basic equations coupling E and B to charges.

A better proposal is to include only fundamental electromagnetic fields on the t-shirt. In addition, we should note that the units of electromagnetic fields were originated by humans and that their choices, and not any fundamental considerations, are the source of the abominable constants ϵ_0 and μ_0. God might have preferred units in which these are set to 1 (and also the speed of light $c = 1$). (The authors of the CGS system of units consider 4π to be 'God-given'. This is a matter of opinion.) Then Maxwell's equations become:

$$\vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \qquad \vec{\nabla} \times \vec{B} = \frac{\partial}{\partial t}\vec{E} + \vec{j} \tag{2}$$

Should we be satisfied, though, to write Maxwell's equations in terms of E and B? In special relativity, these elements are unified in the field tensor $F^{\mu\nu}$. Using $F^{\mu\nu}$, there are two field equations

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0 \tag{3}$$

Mathematical physicists generally agree on the simplifications up to this point. But, if we wish to take a further step, there are two divergent philosophies. I will call these the Weinberg and Wheeler approaches, following the very different philosophies of these authors' textbooks on general relativity.

The Wheeler approach is to abstract the electromagnetic field tensor as a differential form (2-form)

$$F = \frac{1}{2} dx^\mu \wedge dx^\nu \; F_{\mu\nu} \tag{4}$$

The differential form is defined in affine geometry, without coordinates, indices, or even a metric. Forms are acted on by a differential operator d, the exterior derivative. For Maxwell's equations, we will also need the dual of the form, and it should be noted that this dual form ${}^*F$ is defined from F using a metric:

$$ {}^*F = \frac{1}{4} dx^\mu \wedge dx^\nu \ \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} \tag{5} $$

In any case, the differential form for F obeys the equations

$$ dF = 0 \qquad d^*F = j \ , \tag{6} $$

where I have written the electromagnetic current as a 1-form.

The Weinberg approach is to look at electromagnetism primarily as a theory of a massless spin-1 field A_μ. To define this field in the quantum theory, the equations for A_μ must respect a local gauge symmetry

$$ A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \tag{7} $$

The simplest equation of motion for A_μ which is explicitly gauge-invariant is found by defining

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{8} $$

Then the second equation in (3) is an integrability condition implying that F can be written in terms of A, and the first equation in (3) is the additional content of Maxwell's equations. Alternatively, we could replace F by A in this equation and find

$$ \partial^2 A_\mu = j_\mu \qquad A_\mu \cong A_\mu + \partial_\mu \alpha \tag{9} $$

Both of these ideas are implicit in the description of the electromagnetic field as a massless field of string theory. On one hand, string theory reproduces the structure of differential forms in the low-energy equations valid for momentum much less than the string tension. On the other hand, string theory has a multi-parameter gauge invariance that subsumes the gauge symmetry of electromagnetism. In fact, this gauge invariance should be considered a special case in which the compactification of the string theory leads to a 4-dimensional theory with a gauge group G that contains a $U(1)$ factor. There is a restriction that the charges and currents to which the A field couples must themselves be strings. In string theory, there is literally nothing but string.

So, my final proposal for the t-shirt is the following:

> And God said:
>
> String theory with $d = 4$, $G \supset U(1)$
>
> and there was light!

God, being God, knows how to remove the restrictive clause. I'm sorry, but I do not.